

# APERTURE OF PLANE CURVES

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ABSTRACT. For any given  $C^\infty$  immersion  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  such that the set  $\mathcal{NS}_{\mathbf{r}} = \mathbb{R}^2 - \bigcup_{s \in S^1} (\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1)))$  is not empty, a simple geometric model of crystal growth is constructed. It is shown that our geometric model of crystal growth never formulates a polygon while it is growing. Moreover, it is shown also that our model always dissolves to a point.

## 1. INTRODUCTION

Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion such that the set

$$(1.1) \quad \mathbb{R}^2 - \bigcup_{s \in S^1} (\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1)))$$

is not the empty set, where  $T_{\mathbf{r}(s)}\mathbb{R}^2$  is identified with  $\mathbb{R}^2$ . The perspective projection of the given plane curve  $\mathbf{r}(S^1)$  from any point of (1.1) does not give the silhouette of  $\mathbf{r}(S^1)$  because it is non-singular. By this reason, the set (1.1) is called the *no-silhouette* of  $\mathbf{r}$  and is denoted by  $\mathcal{NS}_{\mathbf{r}}$  (see Figure 1). The notion of no-silhouette

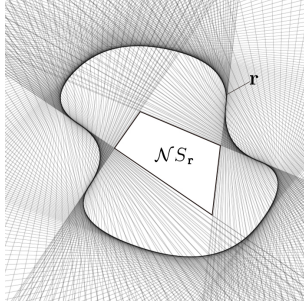


FIGURE 1. The no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ .

was first defined and studied from the viewpoint of perspective projection in [10]. In [11] it has been shown that the topological closure of no-silhouette is a Wulff shape, which is the well-known geometric model of crystal at equilibrium introduced by G. Wulff in [14].

In this paper, we show that by rotating all tangent lines about their tangent points simultaneously with the same angle, we always obtain a geometric model of crystal growth (Proposition 6), our model never formulates a polygon while it is growing (Theorem 1), our model always dissolves to a point (Theorems 2), and

1991 *Mathematics Subject Classification.* 58K30, 68T45, 82D25.

*Key words and phrases.* aperture, aperture angle, aperture point, Wulff shape.

our model is growing in a relatively simple way when the given  $\mathbf{r}$  has no inflection points (Theorem 3).

For any  $C^\infty$  immersion  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  and any real number  $\theta$ , define the new set

$$\mathcal{NS}_{\theta, \mathbf{r}} = \mathbb{R}^2 - \bigcup_{s \in S^1} (\mathbf{r}(s) + R_\theta(d\mathbf{r}_s(T_s(S^1)))) ,$$

where  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the rotation defined by  $R_\theta(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$  (see Figure 2). When the given  $\mathbf{r}$  has its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ , by definition,

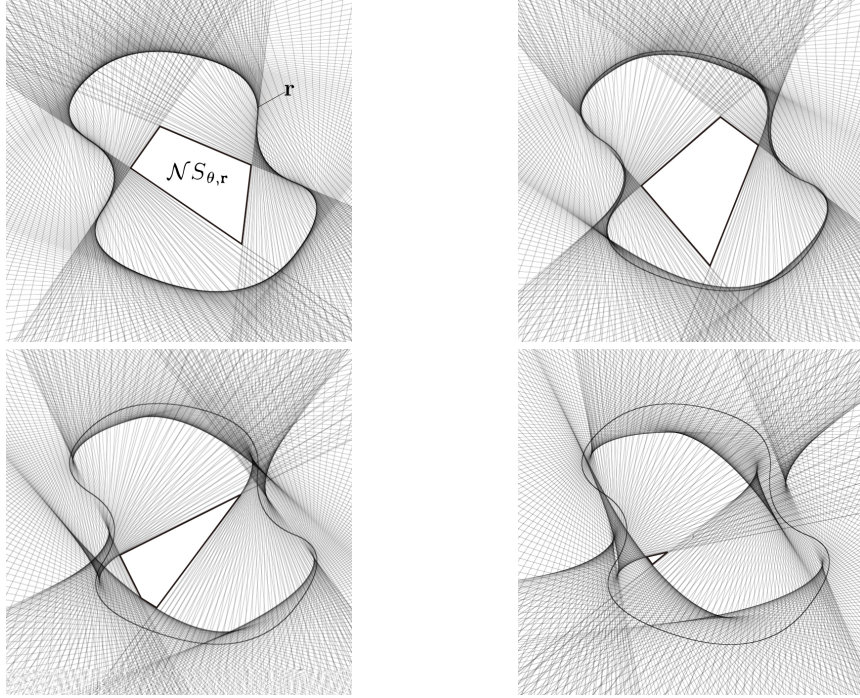


FIGURE 2.  $\mathcal{NS}_{\theta, \mathbf{r}}$  for several  $\theta$ s. Left top :  $\theta = 0$ , right top :  $\theta = \pi/12$ , left bottom :  $\theta = \pi/6$ , right bottom :  $\theta = \pi/4$ .

it follows that  $\mathcal{NS}_{\mathbf{r}} = \mathcal{NS}_{0, \mathbf{r}}$ .

**Lemma 1.1.** *For any  $C^\infty$  immersion  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$ ,  $\mathcal{NS}_{\frac{\pi}{2}, \mathbf{r}}$  is the empty set.*

Proof of Lemma 1.1 For any point  $P \in \mathbb{R}^2$ , let  $F_P : S^1 \rightarrow \mathbb{R}$  be the function defined by

$$(1.2) \quad F_P(s) = (P - \mathbf{r}(s)) \cdot (P - \mathbf{r}(s)),$$

where the dot in the center stands for the scalar product of two vectors. Since  $F_P$  is a  $C^\infty$  function and  $S^1$  is compact, there exist the maximum and the minimum of the set of images  $\{F_P(s) \mid s \in S^1\}$ . Let  $s_1$  (resp.,  $s_2$ ) be a point of  $S^1$  at which  $F_P$  attains its maximum (resp., minimum). Then, both  $s_1$  and  $s_2$  are critical points of  $F_P$ . Thus, differentiating (1.2) with respect to  $s$  yields that the vector  $(P - \mathbf{r}(s_i))$  is perpendicular to the tangent line to  $\mathbf{r}$  at  $\mathbf{r}(s_i)$ . It follows that  $P \in (\mathbf{r}(s_i) + R_{\frac{\pi}{2}}(d\mathbf{r}_{s_i}(T_{s_i}(S^1)))$ .  $\square$

In Section 2, it turns out that with respect to the Pompeiu-Hausdorff metric the topological closure of  $\mathcal{NS}_{\theta, \mathbf{r}}$  varies continuously depending on  $\theta$  while  $\mathcal{NS}_{\theta, \mathbf{r}}$  is not empty (Proposition 7). Therefore, by Lemma 1.1, the following notion of aperture angle  $\theta_{\mathbf{r}}$  ( $0 < \theta_{\mathbf{r}} \leq \frac{\pi}{2}$ ) is well-defined.

**Definition 1.** Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then,  $\theta_{\mathbf{r}}$  ( $0 < \theta_{\mathbf{r}} \leq \frac{\pi}{2}$ ) is defined as the largest angle which satisfies  $\mathcal{NS}_{\theta, \mathbf{r}} \neq \emptyset$  for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ). The angle  $\theta_{\mathbf{r}}$  is called the *aperture angle* of the given  $\mathbf{r}$ .

In Section 2, it turns out also that  $\overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  is a Wulff shape for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ), where  $\overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  stands for the topological closure of  $\mathcal{NS}_{\theta, \mathbf{r}}$  (Proposition 6). We are interested in how the Wulff shape  $\overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  dissolves as  $\theta$  goes to  $\theta_{\mathbf{r}}$  from 0.

**Theorem 1.** Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then, for any  $\theta$  ( $0 < \theta < \theta_{\mathbf{r}}$ ),  $\overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  is never a polygon even if the given  $\overline{\mathcal{NS}_{\mathbf{r}}}$  is a polygon.

By Theorem 1, none of  $\overline{\mathcal{NS}_{\frac{\pi}{12}, \mathbf{r}}}$ ,  $\overline{\mathcal{NS}_{\frac{\pi}{6}, \mathbf{r}}}$  and  $\overline{\mathcal{NS}_{\frac{\pi}{4}, \mathbf{r}}}$  in Figure 2 is a polygon although  $\overline{\mathcal{NS}_{0, \mathbf{r}}}$  is a polygon constructed by four tangent lines to  $\mathbf{r}$  at four inflection points.

**Theorem 2.** Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then, there exists the unique point  $P_{\mathbf{r}} \in \mathbb{R}^2$  such that for any sequence  $\{\theta_i\}_{i=1,2,\dots} \subset [0, \theta_{\mathbf{r}})$  satisfying  $\lim_{i \rightarrow \infty} \theta_i = \theta_{\mathbf{r}}$ , the following holds: .

$$\lim_{i \rightarrow \infty} d_H(\overline{\mathcal{NS}_{\theta_i, \mathbf{r}}}, P_{\mathbf{r}}) = 0.$$

Here,  $d_H : \mathcal{H}(\mathbb{R}^2) \times \mathcal{H}(\mathbb{R}^2) \rightarrow \mathbb{R}$  is the Pompeiu-Hausdorff metric (for the Pompeiu-Hausdorff metric, see Section 2). Theorem 2 justifies the following definition.

**Definition 2.** Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then, the set  $\cup_{\theta \in [0, \theta_{\mathbf{r}})} \overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  is called the *aperture* of  $\mathbf{r}$  and the unique point  $P_{\mathbf{r}} = \lim_{\theta \rightarrow \theta_{\mathbf{r}}} \overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  is called the *aperture point* of  $\mathbf{r}$ . Here,  $\theta_{\mathbf{r}}$  ( $0 < \theta_{\mathbf{r}} \leq \frac{\pi}{2}$ ) is the aperture angle of  $\mathbf{r}$ .

The simplest example is a circle. The aperture of a circle is the topological closure of its inside region and the aperture point of it is its center. In this case, the aperture angle is  $\pi/2$ . In general, in the case of curves with no inflection points, the crystal growth is relatively simpler than in the case of curves with inflections as follows.

**Theorem 3.** Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Suppose that  $\mathbf{r}$  has no inflection points. Then, for any two  $\theta_1, \theta_2$  satisfying  $0 \leq \theta_1 < \theta_2 < \theta_{\mathbf{r}}$ , the following inclusion holds:

$$\mathcal{NS}_{\theta_1, \mathbf{r}} \supset \mathcal{NS}_{\theta_2, \mathbf{r}}.$$

Figure 2 shows that in general it is impossible to expect the same property for a curve with inflection points.

In Section 2, preliminaries are given. Theorems 1, 2 and 3 are proved in Sections 3, 4 and 5 respectively.

## 2. PRELIMINARIES

**2.1. Spherical curves.** Let  $\tilde{\mathbf{r}} : S^1 \rightarrow S^2$  be a  $C^\infty$  immersion. Let  $\tilde{\mathbf{t}} : S^1 \rightarrow S^2$  be the mapping defined by

$$\tilde{\mathbf{t}}(s) = \frac{\tilde{\mathbf{r}}'(s)}{\|\tilde{\mathbf{r}}'(s)\|},$$

where  $\tilde{\mathbf{r}}'(s)$  stands for differentiating  $\tilde{\mathbf{r}}(s)$  with respect to  $s \in S^1$ . Let  $\tilde{\mathbf{n}} : S^1 \rightarrow S^2$  be the mapping defined by

$$\det(\tilde{\mathbf{r}}(s), \tilde{\mathbf{t}}(s), \tilde{\mathbf{n}}(s)) = 1.$$

The mapping  $\tilde{\mathbf{n}} : S^1 \rightarrow S^2$  is called the *spherical dual of  $\tilde{\mathbf{r}}$* . The singularities of  $\tilde{\mathbf{n}}$  belong to the class of Legendrian singularities which are relatively well-investigated (for instance, see [1, 2, 3]). Let  $U$  be an open arc of  $S^1$ . Suppose that  $\|\tilde{\mathbf{r}}'(s)\| = 1$  for any  $s \in U$ . Then, for the orthogonal moving frame  $\{\mathbf{r}(s), \mathbf{t}(s), \mathbf{n}(s)\}$ , ( $s \in U$ ), the following Serre-Frenet type formula has been known.

**Lemma 2.1** ([7, 8]).

$$\begin{cases} \tilde{\mathbf{r}}'(s) &= \tilde{\mathbf{t}}(s) \\ \tilde{\mathbf{t}}'(s) &= -\tilde{\mathbf{r}}(s) + \kappa_g(\theta)\tilde{\mathbf{n}}(s) \\ \tilde{\mathbf{n}}'(s) &= -\kappa_g(\theta)\tilde{\mathbf{t}}(s). \end{cases}$$

Here,  $\kappa_g(\theta)$  is defined by

$$\kappa_g(\theta) = \det(\tilde{\mathbf{r}}(s), \tilde{\mathbf{t}}(s), \tilde{\mathbf{t}}'(s)).$$

Let  $N$  be the north pole  $(0, 0, 1)$  of the unit sphere  $S^2 \subset \mathbb{R}^3$  and let  $S_{N,+}^2$  be the northern hemisphere  $\{P \in S^2 \mid N \cdot P > 0\}$ , where  $N \cdot P$  stands for the scalar product of two vectors  $N, P \in \mathbb{R}^3$ . Then, define the mapping  $\alpha_N : S_{N,+}^2 \rightarrow \mathbb{R}^2 \times \{1\}$ , which is called the *central projection*, as follows:

$$\alpha_N(P_1, P_2, P_3) = \left( \frac{P_1}{P_3}, \frac{P_2}{P_3}, 1 \right),$$

where  $P = (P_1, P_2, P_3) \in S_{N,+}^2$ . Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion. Then, from  $\mathbf{r}$  we can naturally obtain a spherical curve  $\tilde{\mathbf{r}} : S^1 \rightarrow S^2$  as follows:

$$\tilde{\mathbf{r}} = \alpha_N^{-1} \circ Id \circ \mathbf{r},$$

where  $Id : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \{1\}$  is the mapping defined by  $Id(P) = (P, 1)$ . For any  $s \in S^1$ , let  $GC_{\tilde{\mathbf{r}}(s)}$  be the intersection  $(\mathbb{R}\tilde{\mathbf{r}}(s) + \mathbb{R}\tilde{\mathbf{t}}(s)) \cap S^2$ . The following clearly holds:

**Lemma 2.2.** *By the central projection  $\alpha_N : S_{N,+}^2 \rightarrow \mathbb{R}^2 \times \{1\}$ ,  $GC_{\tilde{\mathbf{r}}(s)} \cap S_{N,+}^2$  is mapped to the line  $\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1))$ .*

One of the merit of considering inside the sphere  $S^2$  is the following:

**Lemma 2.3** ([10]). *Let  $\tilde{\mathbf{r}} : S^1 \rightarrow S^2$  be a Legendrian mapping. Then, the following two are equivalent conditions.*

(1) *The set*

$$S^2 - \bigcup_{s \in S^2} GC_{\tilde{\mathbf{r}}(s)}$$

*is not empty and  $N$  is inside this open set.*

- (2) The connected subset  $\{\tilde{\mathbf{n}}(s) \mid s \in S^1\}$  is inside  $S_{N,+}^2$ , where  $\tilde{\mathbf{n}}$  is the dual of  $\tilde{\mathbf{r}}$ .

Let  $\Psi_N : S^2 - \{\pm N\} \rightarrow S^2$  be the mapping defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}}(N - (N \cdot P)P).$$

The mapping  $\Psi_N$  is very useful for studying spherical pedals, pedal unfoldings of spherical pedals, hedgehogs, and Wulff shapes (see [7, 8, 9, 10, 11]). There is also a hyperbolic version of  $\Psi_N$  ([6]). The fundamental properties of  $\Psi_N$  is as follows:

- (1) For any  $P \in S^2 - \{\pm N\}$ , the equality  $P \cdot \Psi_N(P) = 0$  holds,
- (2) for any  $P \in S^2 - \{\pm N\}$ , the property  $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$  holds,
- (3) for any  $P \in S^2 - \{\pm N\}$ , the property  $N \cdot \Psi_N(P) > 0$  holds,
- (4) the restriction  $\Psi_N|_{S_{N,+}^2 - \{N\}} : S_{N,+}^2 - \{N\} \rightarrow S_{N,+}^2 - \{N\}$  is a  $C^\infty$  diffeomorphism.

By these properties, we have the following:

**Lemma 2.4.** *The mapping  $\alpha_N \circ \Psi_N \circ \alpha_N^{-1} : \mathbb{R}^2 \times \{1\} - \{N\} \rightarrow \mathbb{R}^2 \times \{1\} - \{N\}$  is the inversion of  $\mathbb{R}^2 \times \{1\} - \{N\}$  with respect to  $N$ .*

**2.2. Spherical polar sets and the spherical polar transform.** For any point  $P$  of  $S^2$ , we let  $H(P)$  be the following set:

$$H(P) = \{Q \in S^2 \mid P \cdot Q \geq 0\}.$$

Here, the dot in the center stands for the scalar product of  $P, Q \in \mathbb{R}^3$ .

**Definition 3** ([11]). Let  $W$  be a subset of  $S^2$ . Then, the set

$$\bigcap_{P \in W} H(P)$$

is called the *spherical polar set* of  $W$  and is denoted by  $W^\circ$ .

Figure 3 illustrates Definition 3. It is clear that  $W^\circ = \bigcap_{P \in W} H(P)$  is closed for any  $W \subset S^2$ .

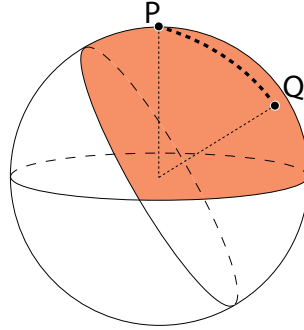


FIGURE 3. Spherical polar set  $\{P, Q\}^\circ = (PQ)^\circ$ .

**Definition 4** ([11]). A subset  $W \subset S^2$  is said to be *hemispherical* if there exists a point  $P \in S^2$  such that  $H(P) \cap W = \emptyset$ .

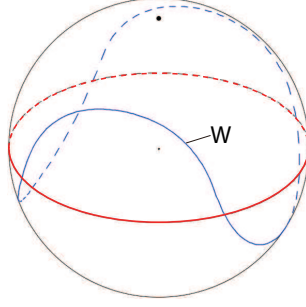
FIGURE 4. Not hemispherical  $W \subset S^2$ .

Figure 4 illustrates Definition 4.

**Definition 5** ([11]). A hemispherical subset  $W \subset S^2$  is said to be *spherical convex* if  $PQ \subset W$  for any  $P, Q \in W$ .

Here,  $PQ$  stands for the following arc:

$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \in S^2 \mid 0 \leq t \leq 1 \right\}.$$

Note that  $\|(1-t)P + tQ\| \neq 0$  for any  $P, Q \in W$  and any  $t \in [0, 1]$  if  $W \subset S^2$  is hemispherical. Note further that  $W^\circ$  is spherical convex if  $W$  is hemispherical and it has an interior point.

**Definition 6** ([11]). Let  $W$  be a hemispherical subset of  $S^2$ . Then, the *spherical convex hull* of  $W$  (denoted by  $\text{s-conv}(W)$ ) is the following set.

$$\text{s-conv}(W) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in W, \sum_{i=1}^k t_i = 1, t_i \geq 0, k \in \mathbb{N} \right\}.$$

**Lemma 2.5** (Lemma 2.5 of [11]). *For any hemispherical finite subset  $W = \{P_1, \dots, P_k\} \subset S^{n+1}$ , the following holds:*

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \mid P_i \in W, \sum_{i=1}^k t_i = 1, t_i \geq 0 \right\}^\circ = H(P_1) \cap \dots \cap H(P_k).$$

Lemma 2.5 is called *Maehara's lemma* (see [11]).

**Definition 7** ([4]). Let  $(X, d)$  be a complete metric space.

- (1) Let  $x$  be a point of  $X$  and let  $B$  a non-empty compact subset of  $X$ . Define

$$d(x, B) = \min\{d(x, y) \mid y \in B\}.$$

Then,  $d(x, B)$  is called the *distance from the point  $x$  to the set  $B$* .

- (2) Let  $A, B$  be two non-empty compact subsets of  $X$ . Define

$$d(A, B) = \max\{d(x, B) \mid x \in A\}.$$

Then,  $d(A, B)$  is called the *distance from the set  $A$  to the set  $B$* .

(3) Let  $A, B$  be two non-empty compact subsets of  $X$ . Define

$$d_H(A, B) = \max\{d(A, B), d(B, A)\}.$$

Then,  $d_H(A, B)$  is called the *Pompeiu-Hausdorff distance between  $A$  and  $B$* .

Let  $(X, d)$  be a complete metric space. The set consisting of non-empty compact subsets of  $X$  is denoted by  $\mathcal{H}(X)$ , which is the metric space with respect to the Pompeiu-Hausdorff metric  $d_H : \mathcal{H}(X) \times \mathcal{H}(X) \rightarrow \mathbb{R}_+ \cup \{0\}$ , where  $d_H$  is the metric naturally induced by the Pompeiu-Hausdorff distance. It is well-known also that the metric space  $(\mathcal{H}(X), d_H)$  is complete. For more details on the complete metric space  $(\mathcal{H}(X), d_H)$ , see for instance [4, 5].

**Definition 8.** Let  $\bigcirc : \mathcal{H}(S^2) \rightarrow \mathcal{H}(S^2)$  be the mapping defined by

$$\bigcirc(A) = A^\circ.$$

The mapping  $\bigcirc : \mathcal{H}(S^2) \rightarrow \mathcal{H}(S^2)$  is called the *spherical polar transform*.

**Proposition 1.** *The spherical polar transform is continuous with respect to the Pompeiu-Hausdorff metric.*

Proof of Proposition 1 Let  $\{A_i\}_{i=1,2,\dots} \subset \mathcal{H}(S^2)$  be a convergent sequence. Set  $A = \lim_{i \rightarrow \infty} A_i$ . In order to prove Proposition 1, it is sufficient to show that  $A^\circ = \lim_{i \rightarrow \infty} A_i^\circ$ .

Suppose that there exists a real number  $\varepsilon > 0$  such that for any  $n \in \mathbb{N}$  there exists an  $i_n$  ( $i_n > n$ ) such that  $d_H(A_{i_n}^\circ, A^\circ) > \varepsilon$ . Then, by Definition 7, it follows that for any  $n \in \mathbb{N}$ , at least one of  $d(A_{i_n}^\circ, A^\circ) > \varepsilon$  and  $d(A^\circ, A_{i_n}^\circ) > \varepsilon$  holds. By taking a subsequence if necessary, from the first we may assume that one of the following holds:

- (1)  $d(A_{i_n}^\circ, A^\circ) > \varepsilon$  for any  $n \in \mathbb{N}$ .
- (2)  $d(A^\circ, A_{i_n}^\circ) > \varepsilon$  for any  $n \in \mathbb{N}$ ,

We first show that (1) implies a contradiction. By Definition 7, it follows that for any  $n \in \mathbb{N}$  there exists a point  $x_n \in A_{i_n}^\circ$  such that  $d(x_n, A^\circ) > \varepsilon$ . Again by Definition 7, it follows that for any  $n \in \mathbb{N}$  there exists a point  $x_n \in A_{i_n}^\circ$  such that the inequality  $d(x_n, y) > \varepsilon$  holds for any  $y \in A^\circ$ . It is known that  $A$  can be characterized as follows ([4]).

$$(2.1) \quad A = \left\{ P \in S^2 \mid \exists P_n \in A_{i_n} (n \in \mathbb{N}) \text{ such that } \lim_{n \rightarrow \infty} P_n = P \right\}.$$

Let  $P$  be a point of  $A$ . By (2.1), for any  $n \in \mathbb{N}$  we may choose a point  $P_n \in A_{i_n}$  such that  $\lim_{n \rightarrow \infty} P_n = P$ . Then, since  $x_n \in A_{i_n}^\circ$ , it follows that  $x_n \cdot P_n \geq 0$ . Since  $S^2$  is compact, there exists a convergent subsequence  $\{x_{j_n}\}_{n=1,2,\dots}$  of the sequence  $\{x_n\}_{n=1,2,\dots}$ . Set  $x = \lim_{n \rightarrow \infty} x_{j_n}$ . Then, the inequality  $d(x_n, y) > \varepsilon$  implies the inequality  $d(x, y) \geq \varepsilon$  for any  $y \in A^\circ$ . On the other hand, the inequality  $x_n \cdot P_n \geq 0$  implies the inequality  $x \cdot P \geq 0$  for any  $P \in A$ . Therefore, the point  $x$  is an element of  $A^\circ$  such that the inequality  $d(x, y) \geq \varepsilon$  holds for any  $y \in A^\circ$ . This is a contradiction.

We next show that (2) implies a contradiction. By the same argument as in (1), we have that for any  $n \in \mathbb{N}$  there exists a point  $x_n \in A^\circ$  such that the inequality  $d(x_n, y_n) > \varepsilon$  for any  $y_n \in A_{i_n}^\circ$ . This implies that there exists an  $M \in \mathbb{N}$  such that for any  $n \in \mathbb{N}$  there exists  $P_n \in A_{i_n}$  such that  $x_n \cdot P_n < -\frac{\varepsilon}{M}$ . Since  $S^2$  is



compact, there exists a subsequence  $\{j_n\}_{n=1,2,\dots}$  of  $\mathbb{N}$  such that both  $\{x_{j_n}\}_{n=1,2,\dots}$ ,  $\{P_{j_n}\}_{n=1,2,\dots}$  are convergent sequences. Set  $x = \lim_{n \rightarrow \infty} x_{j_n}$  and  $P = \lim_{n \rightarrow \infty} P_{j_n}$ . Then, the inequality  $x_n \cdot P_n < -\frac{\varepsilon}{M}$  implies the inequality  $x \cdot P \leq -\frac{\varepsilon}{M}$ . On the other hand, since  $A^\circ$  is compact,  $x$  belongs to  $A^\circ$ . Moreover, by (2.1),  $P$  belongs to  $A$ . Hence, by Definition 3, the scalar product  $x \cdot P$  must be non-negative. Therefore, we have a contradiction.  $\square$

**2.3. Wulff shapes.** Let  $\mathbb{R}_+$  be the set  $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$  and let  $h : S^1 \rightarrow \mathbb{R}_+$  be a continuous function. For any  $s \in S^1 \subset \mathbb{R}^2$ , set

$$\Gamma_{h,s} = \{P \in \mathbb{R}^2 \mid P \cdot s \leq h(s)\},$$

where the dot in the center stands for the scalar product of two vectors  $P, s \in \mathbb{R}^2$ . The following set is called the *Wulff shape associated with the support function  $h$*  (see Figure 5):

$$\mathcal{W}_h = \bigcap_{s \in S^1} \Gamma_{h,s}.$$

For any crystal at equilibrium the shape of it can be constructed as the Wulff

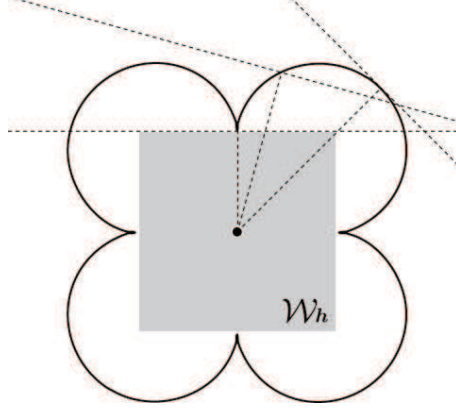


FIGURE 5. The Wulff shape associated with the support function  $h$ .

shape  $\mathcal{W}_h$  by an appropriate support function  $h$  ([14]). It is clear that any Wulff shape  $\mathcal{W}_h$  is a convex body (namely, it is compact, convex and the origin of  $\mathbb{R}^2$  is contained in  $\mathcal{W}_h$  as an interior point). It has been known that its converse, too, holds as follows.

**Proposition 2** (p. 573 of [13]). *Let  $W$  be a subset of  $\mathbb{R}^2$ . Then, there exists a parallel translation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $T(W)$  is the Wulff shape associated with an appropriate support function if and only if  $W$  is a convex body.*

**Proposition 3** (Theorem 1.1 of [11]). *Let  $\{\mathcal{W}_{h_i}\}_{i=1,2,\dots}$  be a Cauchy sequence of Wulff shapes in  $\mathcal{H}_{\text{conv}}(\mathbb{R}^2)$  with respect to the Pompeiu-Hausdorff metric  $d_H$ . Suppose that  $\lim_{i \rightarrow \infty} \mathcal{W}_{h_i}$  does not have an interior point. Then, it must be a point or a segment.*

**Proposition 4** (Theorem 1.2 of [11]). *Let  $h : S^1 \rightarrow \mathbb{R}_+$  be a continuous function. Then, for the Wulff shape  $\mathcal{W}_h$ , the set  $\text{Id}^{-1} \circ \alpha_N \left( (\alpha_N^{-1} \circ \text{Id}(\mathcal{W}_h))^\circ \right)$  is the Wulff shape associated with an appropriate support function.*



The Wulff shape  $Id^{-1} \circ \alpha_N \left( (\alpha_N^{-1} \circ Id(\mathcal{W}_h))^\circ \right)$  is called the *dual Wulff shape* of  $\mathcal{W}_h$ .

**Proposition 5** (Theorem 1.3 of [11]). *Let  $h : S^1 \rightarrow \mathbb{R}_+$  be a function of class  $C^1$ . Then, the Wulff shape  $\mathcal{W}_h$  is never a polygon.*

**Proposition 6.** *Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then, for any  $\theta \in [0, \theta_{\mathbf{r}})$ , there exists a parallel translation  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $T_\theta(\overline{\mathcal{NS}_{\theta, \mathbf{r}}})$  is a Wulff shape  $\mathcal{W}_{h_\theta}$  by an appropriate support function  $h_\theta : S^1 \rightarrow \mathbb{R}_+$ .*

Proof of Proposition 6 We first show that  $\mathcal{NS}_{\theta, \mathbf{r}}$  is an open set for any  $\theta \in [0, \theta_{\mathbf{r}})$ . Let  $P$  be a point of  $\mathcal{NS}_{\theta, \mathbf{r}}$ . Suppose that for any positive integer  $n$ , there exists a point  $P_n \in D(P, \frac{1}{n}) \cap (\cup_{s \in S^1} (\mathbf{r}(s) + R_\theta(\mathbf{dr}_s(T_s(S^1))))$ , where  $D(P, \frac{1}{n})$  is the disc  $D(P, \frac{1}{n}) = \{Q \in \mathbb{R}^2 \mid \|P - Q\| \leq \frac{1}{n}\}$ . Then, since  $S^1$  is compact, by taking a subsequence if necessary, we may assume that there exists a convergent sequence  $s_n \in S^1$  ( $n \in \mathbb{N}$ ) such that  $P_n$  belongs to  $D(P, \frac{1}{n}) \cap (\mathbf{r}(s_n) + R_\theta(\mathbf{dr}_{s_n}(T_{s_n}(S^1))))$ . Then, we have that  $P \in \mathbf{r}(s) + R_\theta(\mathbf{dr}_s(T_s(S^1)))$  where  $s = \lim_{i \rightarrow \infty} s_i$ , which implies  $P \notin \mathcal{NS}_{\theta, \mathbf{r}}$ . Hence,  $\mathcal{NS}_{\theta, \mathbf{r}}$  is an open set.

Since  $\theta < \theta_{\mathbf{r}}$ , it follows that  $\mathcal{NS}_{\theta, \mathbf{r}} \neq \emptyset$ . Let  $P$  be a point of  $\mathcal{NS}_{\theta, \mathbf{r}}$ . Let  $P_s \in \mathbf{r}(s) + R_\theta \mathbf{dr}_s(T_s(S^1))$  be the point such that the vector  $PP_s$  is perpendicular to the line  $\mathbf{r}(s) + R_\theta \mathbf{dr}_s(T_s(S^1))$ . Then, by obtaining the concrete expression of  $P_s$ , it follows that the mapping  $f : S^1 \rightarrow \mathbb{R}^2$  defined by  $f(s) = P_s$  is of class  $C^\infty$ . By Subsection 2.1 and [7], the mapping  $f : S^1 \rightarrow \mathbb{R}^2$  is exactly the pedal curve of the family of lines  $\{\mathbf{r}(s) + R_\theta \mathbf{dr}_s(T_s(S^1))\}_{s \in S^1}$  relative to the pedal point  $P \in \mathcal{NS}_{\theta, \mathbf{r}}$ . Let  $I : \mathbb{R}^2 - \{P\} \rightarrow \mathbb{R}^2 - \{P\}$  be the plane inversion defined by  $I(Q) = P - \frac{1}{\|Q - P\|^2}(Q - P)$ . Since  $P \in \mathcal{NS}_{\theta, \mathbf{r}}$ , the composed mapping  $\mathbf{n} = I \circ f$  is well-defined and of class  $C^\infty$ . The mapping  $\mathbf{n}$  is exactly the dual curve of the family of lines  $\{\mathbf{r}(s) + R_\theta \mathbf{dr}_s(T_s(S^1))\}_{s \in S^1}$  relative to the point  $P \in \mathcal{NS}_{\theta, \mathbf{r}}$ . Let the boundary of convex hull of  $\mathbf{n}(S^1)$  be denoted by  $\partial \text{conv}(\mathbf{n}(S^1))$ . Then, by the construction,  $\partial \text{conv}(\mathbf{n}(S^1))$  intersect the half line  $\{P + \lambda s \mid \lambda \in \mathbb{R}_+\}$  exactly at one point for any  $s \in S^1$ . Thus, the composed image  $I(\partial \text{conv}(\mathbf{n}(S^1)))$  intersect the half line  $\{P + \lambda s \mid \lambda \in \mathbb{R}_+\}$  exactly at one point for any  $s \in S^1$ . Moreover, the intersecting points depend on  $s$  continuously. Hence, by corresponding  $s \in S^1$  to the distance between  $P$  and the unique intersecting point  $I(\partial \text{conv}(\mathbf{n}(S^1))) \cap \{P + \lambda s \mid \lambda \in \mathbb{R}_+\}$ , we obtain the well-defined continuous function  $h_\theta : S^1 \rightarrow \mathbb{R}_+$ . Since  $\mathbf{n}$  is of class  $C^\infty$ , it is easily seen that the obtained function  $h_\theta$  satisfies the assumption of Theorem 6.3 in [11]. Let  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the parallel translation given by  $T_\theta(x, y) = (x, y) - P$ . Then, by Theorem 6.3 of [11], it follows that

$$T_\theta(\overline{\mathcal{NS}_{\theta, \mathbf{r}}}) = \mathcal{W}_{h_\theta}.$$

□

**Proposition 7.** *Let  $\mathbf{r} : S^1 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  immersion with its no-silhouette  $\mathcal{NS}_{\mathbf{r}}$ . Then, the map  $\omega : [0, \theta_{\mathbf{r}}) \rightarrow \mathcal{H}_{\text{conv}}(\mathbb{R}^2)$  defined by  $\omega(\theta) = \overline{\mathcal{NS}_{\theta, \mathbf{r}}}$  is continuous,*

Proof of Proposition 7 Let  $C^0(S^1, \mathbb{R}_+)$  be the set consisting of continuous function  $S^1 \rightarrow \mathbb{R}_+$ . The set  $C^0(S^1, \mathbb{R}_+)$  is a (non-complete) metric space with respect to the metric  $d_{\text{norm}}(h_1, h_2) = \max_{s \in S^1} |h_1(s) - h_2(s)|$ . Let  $\Gamma : [0, \theta_{\mathbf{r}}) \rightarrow C^0(S^1, \mathbb{R}_+)$  (resp.  $\Omega : C^0(S^1, \mathbb{R}_+) \rightarrow \mathcal{H}_{\text{conv}}(\mathbb{R}^2)$ ) be the mapping defined by  $\Gamma(\theta) = h_\theta$  (resp.  $\Omega(h) = \mathcal{W}_h$ ), where  $h_\theta$  is the continuous function defined in the proof of

Proposition 6. Then, in order to show that  $\omega$  is continuous, it is sufficient to show that both  $\Gamma, \Omega$  are continuous.

We first show that  $\Gamma$  is continuous. Let  $\tilde{h} : S^1 \rightarrow \mathbb{R}_+$  be the function defined by

$$\tilde{h}(\cos \lambda, \sin \lambda) = \|P - I(\partial \text{conv}(\mathbf{n}(S^1))) \cap \{P + t(\cos \lambda, \sin \lambda) \mid t \in \mathbb{R}_+\}\|,$$

where the set  $I(\partial \text{conv}(\mathbf{n}(S^1))) \cap \{P + t(\cos \lambda, \sin \lambda) \mid t \in \mathbb{R}_+\}$ , which appeared in the proof of Proposition 6, is a one point set and it is regarded as a point. By obtaining the concrete expression of  $\mathbf{n}$  given in the proof of Proposition 6, it is easily seen that  $\mathbf{n}$  is smoothly depending on  $\theta \in [0, \theta_r)$ . Thus,  $\tilde{h}$  is continuously depending on  $\theta \in [0, \theta_r)$ . Since  $I$  is a  $C^\infty$  diffeomorphism of  $\mathbb{R}^2 - \{P\}$ , it follows that  $h_\theta$  is continuously depending on  $\theta \in [0, \theta_r)$ . Hence,  $\Gamma$  is a continuous mapping.

We next show that  $\Omega$  is continuous. Let  $\{h_i\}_{i=1,2,\dots} \subset C^0(S^1, \mathbb{R}_+)$  be a convergent sequence to an element of  $C^0(S^1, \mathbb{R}_+)$ . Set  $h = \lim_{i \rightarrow \infty} h_i$ . We also set

$$W = \left\{ P \in \mathbb{R}^2 \mid \exists P_i \in \mathcal{W}_{h_i} (i \in \mathbb{N}); \lim_{i \rightarrow \infty} P_i = P \right\}.$$

Then, it is easily seen that  $\mathbb{R}^2 - W$  is an open set. Thus,  $W$  is a closed set.

We show  $\mathcal{W}_h = W$ . Let  $P$  be an interior point of  $\mathcal{W}_h$ . Then, since  $h = \lim_{i \rightarrow \infty} h_i$ ,  $P$  must be an interior point of  $\mathcal{W}_{h_i}$  for any sufficiently large  $i$ . Thus,  $P$  is contained in  $W$ . Since both  $\mathcal{W}_h$  and  $W$  are closed, it follows that  $\mathcal{W}_h \subset W$ . Next, Let  $Q$  be a point of  $W$ . Suppose that  $Q$  is not contained in  $\mathcal{W}_h$ . Then, there exists  $s_0 \in S^1$  such that  $(Q \cdot s_0) > h(s_0)$ , where  $(Q \cdot s_0)$  stands for the scalar product of two vectors  $Q, s_0 \in \mathbb{R}^2$ . Set  $\varepsilon = (Q \cdot s_0) - h(s_0) > 0$ . Since  $h = \lim_{i \rightarrow \infty} h_i$ , it follows that  $(Q \cdot s_0) - h_i(s_0) > \frac{\varepsilon}{2}$  for any sufficiently large  $i$ . This contradicts to the assumption that  $Q \in W$ . Hence, we have that  $W \subset \mathcal{W}_h$ , and it follows that  $\mathcal{W}_h = W$ .

The remaining part of the proof that  $\Omega$  is continuous is to show the following:

$$(2.2) \quad \lim_{i \rightarrow \infty} d_H(W, \mathcal{W}_{h_i}) = 0.$$

In order to show (2.2), by the construction of  $W$ , it is sufficient to show that  $\{\mathcal{W}_{h_i}\}_{i=1,2,\dots}$  is a Cauchy sequence of  $\mathcal{H}(\mathbb{R}^2)$ . Since  $\{h_i\}_{i=1,2,\dots}$  is a Cauchy sequence of  $C^0(S^1, \mathbb{R}_+)$ , it is clear that  $\{\mathcal{W}_{h_i}\}_{i=1,2,\dots}$  is a Cauchy sequence. Therefore, we have that  $\lim_{i \rightarrow \infty} d_H(W, \mathcal{W}_{h_i}) = 0$  and it follows that  $\Omega$  is continuous.  $\square$

### 3. PROOF OF THEOREM 1

By Proposition 6, there exists a parallel translation  $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $T_\theta(\overline{\mathcal{NS}_{\theta, \mathbf{r}}})$  is a Wulff shape. In particular,  $T_\theta(\overline{\mathcal{NS}_{\theta, \mathbf{r}}})$  contains the origin as an interior point. Set  $\tilde{\mathbf{r}} = \alpha_N^{-1} \circ Id \circ T_\theta \circ \mathbf{r}$  and  $\tilde{\mathbf{n}}_\theta(s) = \cos \theta \tilde{\mathbf{n}}(s) - \sin \theta \tilde{\mathbf{t}}(s)$  for  $s \in S^1$ . We investigate the singularities of  $\tilde{\mathbf{n}}_\theta$ . Let  $U$  be an open arc of  $S^1$ . By using the arc-length parameter of  $\tilde{\mathbf{r}}|_U$ , without loss of generality, from the first we may assume that  $\|\tilde{\mathbf{r}}'(s)\| = 1$  for  $s \in U$ . Then, by Lemma 2.1, we have the following:

$$\tilde{\mathbf{n}}'_\theta(s) = -\kappa_g(s) \cos \theta \tilde{\mathbf{t}}(s) + \sin \theta \tilde{\mathbf{r}}(s) - \kappa_g(s) \sin \theta \tilde{\mathbf{n}}(s).$$

Since the angle  $\theta$  satisfies  $0 < \theta < \theta_r \leq \frac{\pi}{2}$  in Theorem 1, it follows that  $\sin \theta \neq 0$ . Therefore,  $\tilde{\mathbf{n}}_\theta$  is non-singular even at the point  $s \in S^1$  such that  $\kappa_g(s) = 0$ .

Next, we show that  $\tilde{\mathbf{n}}_\theta(s) \cdot N > 0$  for any  $s \in S^1$ . Let the dual of  $\tilde{\mathbf{n}}_\theta$  be denoted by  $\tilde{\mathbf{r}}_\theta$ . Then, it follows that  $\tilde{\mathbf{r}}_\theta$  is a Legendrian mapping and the following equality holds.

$$S_{N,+}^2 \cap \left( S^2 - \bigcup_{s \in S^1} GH_{\tilde{\mathbf{r}}_\theta} \right) = \alpha_N^{-1} \circ Id \circ \mathcal{NS}_{\theta,\mathbf{r}}.$$

Since  $\theta < \theta_{\mathbf{r}}$ , by Lemma 2.3, we have that  $\tilde{\mathbf{n}}_\theta(s) \cdot N > 0$  for any  $s \in S^1$ . Thus, the spherical convex hull of  $\{\tilde{\mathbf{n}}_\theta(s) \mid s \in S^1\}$  is well-defined. Since  $\tilde{\mathbf{n}}_\theta$  is non-singular, the boundary of  $\text{s-conv}(\{\tilde{\mathbf{n}}_\theta(s) \mid s \in S^1\})$  is a submanifold of class  $C^1$  (for instance see [12, 15]). By the property (4) of  $\Psi_N$ , the boundary of  $\Psi_N(\text{s-conv}(\{\tilde{\mathbf{n}}_\theta(s) \mid s \in S^1\}))$  is a submanifold of class  $C^1$ . It follows that the boundary of  $Id^{-1} \circ \alpha_N \circ \Psi_N(\text{s-conv}(\{\tilde{\mathbf{n}}_\theta(s) \mid s \in S^1\}))$  is a submanifold of class  $C^1$ .

On the other hand, by constructions, it follows that  $T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}})$  is a Wulff shape  $\mathcal{W}_h$  with the support function  $h$  whose graph with respect to the polar coordinate expression is the boundary of  $Id^{-1} \circ \alpha_N \circ \Psi_N(\text{s-conv}(\{\tilde{\mathbf{n}}_\theta(s) \mid s \in S^1\}))$ .

Therefore, the support function  $h$  for the Wulff shape  $T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}})$  is of class  $C^1$  and it follows that  $\mathcal{W}_h$  is never a polygon by Proposition 5.  $\square$

#### 4. PROOF OF THEOREM 2

By Proposition 6, for any  $i \in \mathbb{N}$  there exists a parallel translation  $T_{\theta_i} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $T_{\theta_i}(\overline{\mathcal{NS}_{\theta_i,\mathbf{r}}})$  is a Wulff shape  $\mathcal{W}_{h_i}$  by an appropriate support function  $h_i$ . By Proposition 4, for any  $i \in \mathbb{N}$  the set  $Id^{-1} \circ \alpha_N \left( (\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i}))^\circ \right)$  is a Wulff shape too. Thus, by Proposition 2, it follows that both  $\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})$  and  $(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i}))^\circ$  belong to  $\mathcal{H}(S^2)$  for any  $i \in \mathbb{N}$ . Moreover, by Proposition 7, we may assume that  $\{T_{\theta_i}\}_{i=1,2,\dots}$  is a Cauchy sequence. Thus, we may assume that both  $\{\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\}_{i=1,2,\dots}$  and  $\{(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i}))^\circ\}_{i=1,2,\dots}$  are Cauchy sequences.

By Proposition 3,  $\lim_{i \rightarrow \infty} \overline{\mathcal{NS}_{\theta_i,\mathbf{r}}}$  is a point or segment. Suppose that it is a segment. Let  $P_1, P_2 \in S^2$  be two boundary points of this segment. Then, by Proposition 1 and Lemma 2.5, we have the following:

$$\lim_{i \rightarrow \infty} (\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i}))^\circ = H(P_1) \cap H(P_2).$$

Let  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}} : S^1 \rightarrow S^2$  be the  $C^\infty$  mapping defined by  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s) = \cos \theta_{\mathbf{r}} \tilde{\mathbf{n}}(s) - \sin \theta_{\mathbf{r}} \tilde{\mathbf{t}}(s)$  for any  $s \in S^1$ , where  $\tilde{\mathbf{n}}$  and  $\tilde{\mathbf{t}}$  are the same  $C^\infty$  mapping as in the proof of Theorem 1. Then, notice that  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(S^1) \subset H(P_1) \cap H(P_2)$ . For any  $j$  ( $j = 1, 2$ ), we let the set  $\{Q \in S^2 \mid P_j \cdot Q = 0\}$  be denoted by  $\partial H(P_j)$ . Then, the intersection  $\partial H(P_1) \cap \partial H(P_2)$  consists of two antipodal points  $Q_1, Q_2$ . By Lemma 2.3 and Proposition 2, there exists  $s_1, s_2 \in S^1$  ( $s_1 \neq s_2$ ) such that  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s_1) = Q_1$ ,  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s_2) = Q_2$ .

On the other hand, since  $0 \leq \theta_{\mathbf{r}} \leq \frac{\pi}{2}$ , similarly as in the proof of Theorem 1, it follows that  $\tilde{\mathbf{n}}_{\theta_{\mathbf{r}}}$  is non-singular. Thus, we have a contradiction.  $\square$

#### 5. PROOF OF THEOREM 3

For any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ) and any  $s \in S^1$ , set

$$\ell_{\theta,s} = \mathbf{r}(s) + R_\theta(\mathbf{dr}_s(T_s S^1)).$$

Let  $f_{\theta,s}(x, y)$  be the affine function which define  $\ell_{\theta,s}$ . Set

$$H_{\theta,s}^+ = \{(x, y) \in \mathbb{R}^2 \mid f_{\theta,s}(x, y) > 0\}, \quad H_{\theta,s}^- = \{(x, y) \in \mathbb{R}^2 \mid f_{\theta,s}(x, y) < 0\}.$$

Then, since  $\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$  is a convex body for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ), it follows that one of  $\mathcal{NS}_{\theta,\mathbf{r}} = \cap_{s \in S^1} H_{\theta,s}^+$  or  $\mathcal{NS}_{\theta,\mathbf{r}} = \cap_{s \in S^1} H_{\theta,s}^-$  holds. By Proposition 6, we may assume that the following holds for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ).

$$\mathcal{NS}_{\theta,\mathbf{r}} = \bigcap_{s \in S^1} H_{\theta,s}^+.$$

Since  $\mathbf{r}$  does not have inflection points, it follows that  $\mathcal{NS}_{0,\mathbf{r}}$  contains  $\mathcal{NS}_{\theta,\mathbf{r}}$  for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ). Thus, for any  $\theta$  ( $0 \leq \theta < \theta_{\mathbf{r}}$ ), we have the following:

$$\begin{aligned} \mathcal{NS}_{\theta,\mathbf{r}} &= \mathcal{NS}_{\theta,\mathbf{r}} \cap \mathcal{NS}_{0,\mathbf{r}} \\ &= \left( \bigcap_{s \in S^1} H_{\theta,s}^+ \right) \cap \mathcal{NS}_{0,\mathbf{r}} \\ &= \bigcap_{s \in S^1} \left( H_{\theta,s}^+ \cap \mathcal{NS}_{0,\mathbf{r}} \right). \end{aligned}$$

Since  $\mathbf{r}$  does not have inflection points, we have that  $H_{\theta_1,s}^+ \cap \mathcal{NS}_{0,\mathbf{r}}$  contains  $H_{\theta_2,s}^+ \cap \mathcal{NS}_{0,\mathbf{r}}$  for any two  $\theta_1, \theta_2 \in [0, \theta_{\mathbf{r}})$  satisfying  $0 \leq \theta_1 < \theta_2 < \theta_{\mathbf{r}}$ . It follows that  $\mathcal{NS}_{\theta_1,\mathbf{r}} \supset \mathcal{NS}_{\theta_2,\mathbf{r}}$  if  $0 \leq \theta_1 < \theta_2 < \theta_{\mathbf{r}}$ .  $\square$

#### ACKNOWLEDGEMENTS

The authors would like to thank the referee for careful reading of the first draft of their paper and giving some comments.

T. Nishimura is partially supported by JSPS KAKENHI Grant Number 26610035.

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The image features a complex, abstract geometric pattern. It consists of a series of overlapping, rounded, petal-like shapes that form a larger, irregular central area. The background is a dense, fine-lined grid of light blue and grey lines. The central area is a white square with a black border, containing the text "NS" in a black, serif font.

$NS$